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Thermal entanglement of a three-qubit system in inhomogeneous magnetic fields

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Abstract

The quantum entanglement of the ground state and the thermal entanglement of the mixed thermal state of a three-qubit system with the Heisenberg interaction are investigated in detail in the presence of an inhomogeneous magnetic field. The concurrence of the model is compared with the magnetization, and the threshold temperature of the entanglement is discussed. The effects of the magnetic field and the coupling coefficient in the thermal entanglement are also examined by considering the concurrence of the three-qubit and two-qubit systems. We found that the thermal entanglement of the system exists at temperatures below the threshold temperature even if the ground state is unentangled.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

Thermal entanglement plays a crucial role in quantum mechanics and quantum information theory. Recently, the investigation of the thermal entanglement has emerged in spin systems in the advance of the quantum information. One of the important aims for quantum computing and quantum communication is to find an entangled resource in solid systems at a finite temperature. The Heisenberg spin model is a basic and most extensively used solid state system. It is known that the Heisenberg interaction can be realized also in quantum dots, nuclear spins, cavity QED [1–4]. A theoretical scheme for multipartite entanglement and quantum information processing with trapped ions is proposed by using a single-resonant interaction [5]. The Deutsch–Jozsa algorithm with high fidelity is investigated based on two-atom interaction in a thermal cavity [6]. The Heisenberg Hamiltonian has been used for quantum computation and quantum teleportation processes [7, 8]. The thermal entanglement of the Heisenberg spin model with anisotropic *XY* interaction was investigated both in the absence of the magnetic field [9] and in the presence of the uniform magnetic field [10, 11]. The entanglement based on the Heisenberg *XYZ* model was also investigated subject to the uniform applied field [12]. In inhomogeneous magnetic fields, the thermal entanglement of the two-qubit system was studied within Heisenberg *XXX* interaction [13], *XXZ* interaction [14] and *XYZ* interaction [15]. Most recently, it has been pointed out that the two-qubit system can be entangled in two distinct temperature regions [16]. The generic behaviour of the thermal entanglement as a function of temperature is investigated in [17]. By quantum Monte Carlo simulations, the thermal entanglement of the reduced state of the two nearby qubits in qubit spin chains was discussed [18]. The divergence of the entanglement in the Heisenberg spin model is studied in [19] by pairwise entanglement with the uniform reduced magnetic field. The experimental determination of the quantum entanglement with a single measurement has been given based on the two copies of the quantum state [20].

In this work, the quantum entanglement of a three-qubit system with the Heisenberg interaction is investigated exactly in the presence of inhomogeneous magnetic fields, and the thermal entanglement of the system is studied at finite temperatures. The threshold temperature of the entanglement is extracted, and we also examine the features of the entanglement and the magnetization of the system. Finally, the difference of the entanglement between the three-qubit and two-qubit systems is discussed. It is shown that the magnetic field, together with the coupling strength of the Heisenberg interaction, can control effectively the entanglement of the system. In a three-qubit system, we found that the thermal entanglement exists at finite temperatures below the threshold temperature even though the ground state is unentangled. This paper is set out as follows. In section 2, starting with the calculation of the eigenstates for a three-qubit system, we get the quantum entanglement of the ground states for the Hamiltonian with the Heisenberg interaction subject to an inhomogeneous magnetic field. The symmetry is discussed for the concurrence of the eigenstates and the critical field is given. The ground state of the system varies between the entangled and unentangled states when the nonuniform applied field changes. In section 3, we give the thermal entanglement of the system at finite temperature and compare the concurrence with the magnetization of the system. The threshold temperature is given as a function of the magnetic field and the coupling coefficient. We also discuss the differences in pairwise concurrences for the three-qubit and two-qubit systems. We summarize our results and give the conclusion in the final section.

2. Quantum entanglement in the ground states

A three-qubit system with the Heisenberg interaction is described, in an inhomogeneous magnetic field, by

$$H = \sum_{n=1}^{3} \left(J\sigma_n^x\sigma_{n+1}^x + J\sigma_n^y\sigma_{n+1}^y + J\sigma_n^z\sigma_{n+1}^z + B_n\sigma_n^z \right), \qquad (1)$$

where $B_1 = B + b$, $B_2 = B - b$, $B_3 = B$, $(b \neq 0)$. Here J is the interaction strength and Pauli matrix σ_n satisfies the periodic boundary condition $\sigma_{n+3} = \sigma_n$. B and b describe the magnetic fields along the z direction so that b measures the degree of the inhomogeneity of the applied field at the sites of the first and second spins for a given uniform field B [13, 15].

The qubit spin system may be treated as being in the ground state under an extremely lower temperature. For finding out the eigenstates of the three-qubit system (1), we work in the standard basis for three spins, $|000\rangle$, $|001\rangle$, $|010\rangle$, $|100\rangle$, $|011\rangle$, $|101\rangle$, $|110\rangle$ and $|111\rangle$. By straightforward diagonalization of the Hamiltonian, the energy eigenstates of equation (1) can be written as follows:

$$\begin{aligned}
|\psi_0\rangle &= |000\rangle, \\
|\psi_j\rangle &= A_j|001\rangle + B_j|010\rangle + C_j|100\rangle, \qquad (j = 1, 2, 3) \\
|\psi_k\rangle &= A_k|011\rangle + B_k|101\rangle + C_k|110\rangle, \qquad (k = 4, 5, 6) \\
|\psi_7\rangle &= |111\rangle,
\end{aligned} \tag{2}$$

where

$$\begin{aligned}
A_j &= \frac{1}{2\sqrt{\Theta_j}}[8J^2 - (\rho_j - B + J + 2b)(\rho_j - 2b - J - B)], \\
B_j &= \frac{1}{2\sqrt{\Theta_j}}[(\rho_j - J - B)(\rho_j + J - B + 2b) - 8J^2], \\
C_j &= \frac{2Jb}{\sqrt{\Theta_j}}, \\
\Theta_j &= b\big[2\rho_j(b(2b - B) - 3J(b - 2B)) + 3J^2(18J + 7b) \\
&\qquad + \rho_j^2(b - 6J) + 6J(b + B)(2b - B) + b(2b - B)^2\big],
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
A_k &= \frac{2bJ}{\sqrt{\Theta_k}}, \\
B_k &= \frac{1}{2\sqrt{\Theta_k}}[8J^2 - (\rho_k + B - J)(\rho_k + B - 2b + J)], \\
C_k &= \frac{1}{2\sqrt{\Theta_k}}[(\rho_k + B - 2b + J)(\rho_k + B + 2b - J) - 8J^2], \\
\Theta_k &= b\big[2\rho_k(B(6J + b) - b(3J + 2b)) + 3J^2(7b - 18J) \\
&\qquad + \rho_k^2(b + 6J) + 6J(B + b)(B - 2b) + b(2b - B)^2\big],
\end{aligned} \tag{4}$$

with the corresponding eigenenergies, respectively, by

$$\begin{aligned}
E_0 &= 3J + 3B, \\
E_i &= \rho_i, \qquad (i = 1, 2, 3, 4, 5, 6) \\
E_7 &= 3J - 3B,
\end{aligned} \tag{5}$$

where

$$\rho_1 = -\frac{4}{3}\sqrt{9J^2 + 3b^2}\cos\left(\frac{\pi}{3} - \frac{1}{3}\arccos\frac{8J^3}{\sqrt{\left(4J^2 + \frac{4}{3}b^2\right)^3}}\right) - J + B, \tag{6}$$

$$\rho_2 = \frac{4}{3}\sqrt{9J^2 + 3b^2}\cos\left(\frac{1}{3}\arccos\frac{8J^3}{\sqrt{\left(4J^2 + \frac{4}{3}b^2\right)^3}}\right) - J + B, \tag{7}$$

$$\rho_3 = -\frac{4}{3}\sqrt{9J^2 + 3b^2}\sin\left(\frac{\pi}{6} - \frac{1}{3}\arccos\frac{8J^3}{\sqrt{\left(4J^2 + \frac{4}{3}b^2\right)^3}}\right) - J + B, \tag{8}$$

$$\rho_4 = -\frac{4}{3}\sqrt{9J^2+3b^2}\cos\left(\frac{\pi}{3} - \frac{1}{3}\arccos\frac{8J^3}{\sqrt{\left(4J^2+\frac{4}{3}b^2\right)^3}}\right) - J - B, \tag{9}$$

$$\rho_5 = \frac{4}{3}\sqrt{9J^2+3b^2}\cos\left(\frac{1}{3}\arccos\frac{8J^3}{\sqrt{\left(4J^2+\frac{4}{3}b^2\right)^3}}\right) - J - B, \tag{10}$$

$$\rho_6 = -\frac{4}{3}\sqrt{9J^2+3b^2}\sin\left(\frac{\pi}{6} - \frac{1}{3}\arccos\frac{8J^3}{\sqrt{\left(4J^2+\frac{4}{3}b^2\right)^3}}\right) - J - B. \tag{11}$$

For convenience we shall work in units such that B, b and J are dimensionless throughout this paper.

The quantum entanglement of the eigenstates can be specified by the value of the concurrence [21], in which R matrix is introduced as follows:

$$R = \rho_{12}(\sigma_y \otimes \sigma_y)\rho_{12}^*(\sigma_y \otimes \sigma_y). \tag{12}$$

Here ρ_{12} is the reduced density matrix and the star indicates its complex conjugate. The concurrence C is defined by

$$C = \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}, \tag{13}$$

where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the positive square roots of the eigenvalues of the matrix R in the descending order. Then, the concurrences of the above energy eigenstates are, respectively,

$$C(|\psi_j\rangle) = 2|B_j C_j|, \qquad C(|\psi_k\rangle) = 2|A_k B_k|, \tag{14}$$

where $j = 1, 2, 3$ and $k = 4, 5, 6$. The other two eigenstates, $|\psi_0\rangle$ and $|\psi_7\rangle$, are unentangled. By considering relations (6)–(11), we note that the coefficients A_i, B_i, C_i $(i = 1, 2, 3, 4, 5, 6)$ satisfy the relations

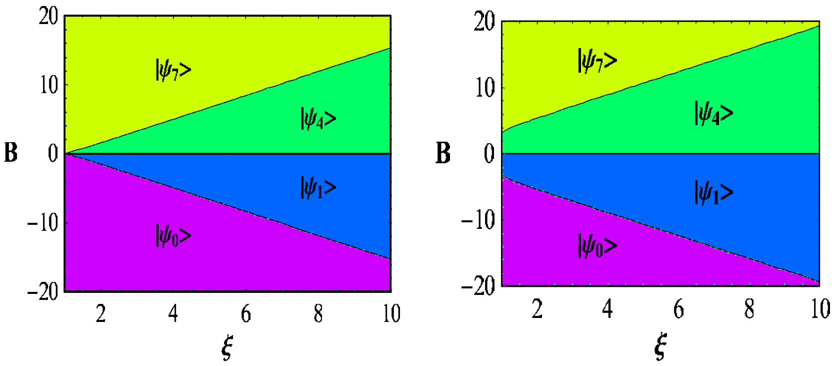
$$A_j^2 = C_{j+3}^2, \qquad B_j^2 = A_{j+3}^2, \qquad C_j^2 = B_{j+3}^2,$$

for $j = 1, 2, 3$. Hence the quantum entanglement of the eigenstates has the symmetrical property

$$C(|\psi_j\rangle) = C(|\psi_{j+3}\rangle), \tag{15}$$

for $j = 1, 2, 3$. By setting

$$\begin{aligned} a_1 &= 1 + 4\xi\cos\left(\frac{\pi}{3} - \frac{1}{3}\arccos\frac{1}{\xi^3}\right), \\ a_2 &= 1 - 4\xi\cos\left(\frac{1}{3}\arccos\frac{1}{\xi^3}\right), \\ a_3 &= 1 + 4\xi\cos\left(\frac{\pi}{3} + \frac{1}{3}\arccos\frac{1}{\xi^3}\right), \\ \xi &= \sqrt{1+\delta^2/3}, \qquad \delta = \frac{b}{J}, \end{aligned} \tag{16}$$



(*a*) (*b*)

Figure 1. The ground states of the three-qubit system, as a function of the inhomogeneity ξ and the magnetic field B: (*a*) the ferromagnetic case; (*b*) the antiferromagnetic case.

the eigenenergy of the Hamiltonian can be written down as

$$\begin{aligned}
&E_0 = 3J + 3B,\\
&E_1 = -Ja_1 + B \quad \text{for} \quad J > 0,\\
&E_1 = -Ja_2 + B \quad \text{for} \quad J < 0,\\
&E_2 = -Ja_1 + B \quad \text{for} \quad J < 0,\\
&E_2 = -Ja_2 + B \quad \text{for} \quad J > 0,\\
&E_3 = -Ja_3 + B,\\
&E_4 = -Ja_1 - B \quad \text{for} \quad J > 0,\\
&E_4 = -Ja_2 - B \quad \text{for} \quad J < 0,\\
&E_5 = -Ja_1 - B \quad \text{for} \quad J < 0,\\
&E_5 = -Ja_2 - B \quad \text{for} \quad J > 0,\\
&E_6 = -Ja_3 - B,\\
&E_7 = 3J - 3B.
\end{aligned} \tag{17}$$

The parameter b, then, δ and ξ denote the inhomogeneity of the applied field. Then we can call them as the zero-field splitting parameters. We are now interested in the quantum entanglement of the ground state of the qubits system. The ground states are indicated in figure 1 as a function of the inhomogeneity parameter δ and magnetic field B for the ferromagnetic ($J < 0$) and antiferromagnetic ($J > 0$) cases, respectively. For the ferromagnetic ($J < 0$), the ground state is $|\psi_4\rangle$ if $0 < B < 2J\left(1 - \xi \cos\left(\frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$, and $|\psi_7\rangle$ if $B > 2J\left(1 - \xi \cos\left(\frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$, respectively, with the critical magnetic field $B_c = 2J\left(1 - \xi \cos\left(\frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$. The ground state is $|\psi_1\rangle$ if $0 > B > -2J\left(1 - \xi \cos\left(\frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$, and $|\psi_0\rangle$ if $B < -2J\left(1 - \xi \cos\left(\frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$ with the critical magnetic field $B_c = -2J\left(1 - \xi \cos\left(\frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$, respectively (see figure 1(*a*)). For the antiferromagnetic case ($J > 0$), the ground state keeps in the state$|\psi_4\rangle$ if $0 < B < 2J\left(1 + \xi \cos\left(\frac{1}{3}\pi - \frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$, and $|\psi_7\rangle$ if $B > 2J\left(1 + \xi \cos\left(\frac{1}{3}\pi - \frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$, respectively, with the critical magnetic field $B_c = 2J\left(1 + \xi \cos\left(\frac{1}{3}\pi - \frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$. And the ground state is $|\psi_1\rangle$ if $0 > B > -2J\left(1 + \xi \cos\left(\frac{1}{3}\pi - \frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$ and $|\psi_0\rangle$ if $B < -2J\left(1 + \xi \cos\left(\frac{1}{3}\pi - \frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$ with the critical magnetic field is $B_c = -2J\left(1 + \xi\cos\left(\frac{1}{3}\pi - \frac{1}{3}\arccos\frac{1}{\xi^3}\right)\right)$, respectively (see figure 1(*b*)).

By introducing the parameters

$$\eta_j = \delta\left(\delta a_j^2 - 6a_j^2 - 4\delta^2 a_j + 6\delta a_j + 54 + 21\delta + 12\delta^2 + 4\delta^3\right),$$

with $j = 1, 2, 3$, and

$$\xi_k = \delta\left(\delta a_k^2 + 6a_k^2 + 6\delta a_k + 4\delta^2 a_k + 21\delta - 54 - 12\delta^2 + 4\delta^3\right),$$

with $k = 1, 2$, the coefficients in the eigenstate $|\psi_1\rangle = A_1|001\rangle + B_1|010\rangle + C_1|100\rangle$ can be written as

$$\begin{aligned} A_1 &= \frac{1}{2\sqrt{\eta_1}}\left(4\delta^2 - a_1^2 + 4\delta + 9\right), \\ B_1 &= \frac{1}{2\sqrt{\eta_1}}\left(a_1^2 - 2\delta a_1 - 2\delta - 9\right), \\ C_1 &= \frac{2\delta}{\sqrt{\eta_1}}, \end{aligned} \tag{18}$$

for $J > 0$, and

$$\begin{aligned} A_1 &= \frac{1}{2\sqrt{\eta_2}}\left(4\delta^2 - a_2^2 + 4\delta + 9\right), \\ B_1 &= \frac{1}{2\sqrt{\eta_2}}\left(a_2^2 - 2\delta a_2 - 2\delta - 9\right), \\ C_1 &= \frac{2\delta}{\sqrt{\eta_2}}, \end{aligned} \tag{19}$$

for $J < 0$. For the eigenstate $|\psi_4\rangle = A_4|011\rangle + B_4|101\rangle + C_4|110\rangle$, we have that

$$\begin{aligned} A_4 &= \frac{2\delta}{\sqrt{\xi_1}}, \\ B_4 &= -\frac{1}{2\sqrt{\xi_1}}\left(a_1^2 + 2\delta a_1 + 2\delta - 9\right), \\ C_4 &= \frac{1}{2\sqrt{\xi_1}}\left(a_1^2 - 4\delta^2 + 4\delta - 9\right), \end{aligned} \tag{20}$$

for $J > 0$, and

$$\begin{aligned} A_4 &= \frac{2\delta}{\sqrt{\xi_2}}, \\ B_4 &= -\frac{1}{2\sqrt{\xi_2}}\left(a_2^2 + 2\delta a_2 + 2\delta - 9\right), \\ C_4 &= \frac{1}{2\sqrt{\xi_2}}\left(a_2^2 - 4\delta^2 + 4\delta - 9\right), \end{aligned} \tag{21}$$

for $J < 0$, respectively. For the energy eigenstates, the quantum entanglement is obtained, respectively, by

$$\begin{aligned} C(\psi_0) &= C(\psi_7) = 0, \\ C(\psi_4) &= C(\psi_1) = 2\left|\frac{\delta}{\eta_2}\left(a_2^2 - 2\delta a_2 - 2\delta - 9\right)\right|, \\ C(\psi_5) &= C(\psi_2) = 2\left|\frac{\delta}{\eta_1}\left(a_1^2 - 2\delta a_1 - 2\delta - 9\right)\right|, \\ C(\psi_6) &= C(\psi_3) = 2\left|\frac{\delta}{\eta_3}\left(a_3^2 - 2\delta a_3 - 9 - 2\delta\right)\right|. \end{aligned} \tag{22}$$

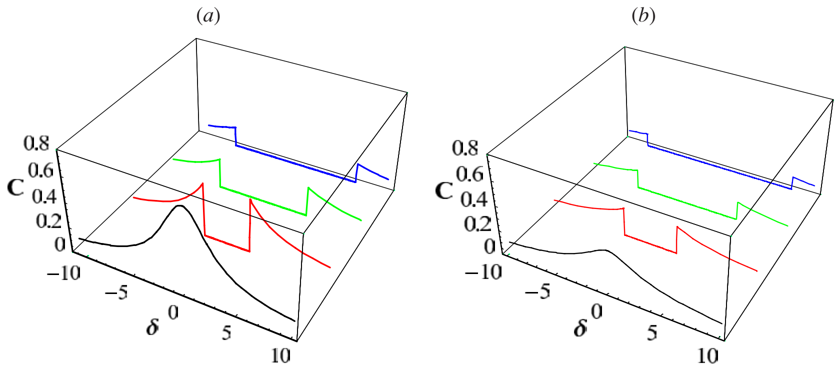


Figure 2. Concurrences of the states versus the zero-field splitting parameter δ for various magnetic fields B: (*a*) $J = -1$ and $B = 0, 1, 3, 5$ from the front to back; (*b*) $J = 1$ and $B = 0, 5, 7.4, 10$ from the front to back.

The quantum entanglement of the ground state is shown in figure 2 for the variation of the concurrence with the zero-field splitting parameter δ for various background applied B fields for the ferromagnetic ($J = -1$) and antiferromagnetic ($J = 1$) cases. The ground state is degenerated as $E_1 = E_4$ when $B = 0$. And the concurrence decreases as the inhomogeneity of the applied field increases both for the ferromagnetic and antiferromagnetic cases. Figure 2(*a*) shows that the ground state is unentangled for $B = 1$ when $\delta < |2.31|$, $B = 3$ when $\delta < |4.65|$ and $B = 5$ when $\delta < |6.76|$, respectively. Otherwise, the ground state is the entangled state $|\psi_4\rangle$. Figure 2(*b*) shows that the ground state is the entangled state $|\psi_4\rangle$ for $B = 5$ when $\delta > |2.58|$, $B = 7.4$ when $\delta > |5.15|$ and $B = 10$ when $\delta > |7.83|$. Otherwise, the ground state is the unentangled state $|\psi_7\rangle$.

3. Thermal entanglement and magnetization

The density operator at the thermal equilibrium is given by $\rho(T) = \mathrm{e}^{-\beta H}/Z$ with the partition function $Z = \mathrm{Tr}\,\mathrm{e}^{-\beta H}$ and $\beta = \frac{1}{kT}$, where k is Boltzmann's constant. At finite temperature, the state of the system may be treated as the mixture of the eigenstates of the energy E_j with the probability $\mathrm{e}^{-\beta E_j}/Z$. The pairwise entanglement of this thermal state can be studied with the reduced density operator similarly as the case of the ground state. By using the scheme denoted by equations (12) and (13), we get that the thermal concurrence of the three-qubit system is of the following form:

$$
\begin{aligned}
C &= \frac{2}{Z}\max\{\sqrt{v_1 v_2} - \sqrt{u_1 u_2}, 0\} \quad \text{for} \quad |w| > \sqrt{v_1 v_2},\\
C &= \frac{2}{Z}\max\{|w| - \sqrt{u_1 u_2}, 0\} \quad \text{for} \quad |w| \leqslant \sqrt{v_1 v_2},
\end{aligned}
\tag{23}
$$

with $Z = u_1 + u_2 + v_1 + v_2$, and

$$
\begin{aligned}
u_1 &= \mathrm{e}^{-\beta E_0} + \sum_{j=1}^{3} \mathrm{e}^{-\beta E_j} A_j^2,\\
u_2 &= \mathrm{e}^{-\beta E_7} + \sum_{j=1}^{3} \mathrm{e}^{-\beta E_{j+3}} A_j^2,
\end{aligned}
$$

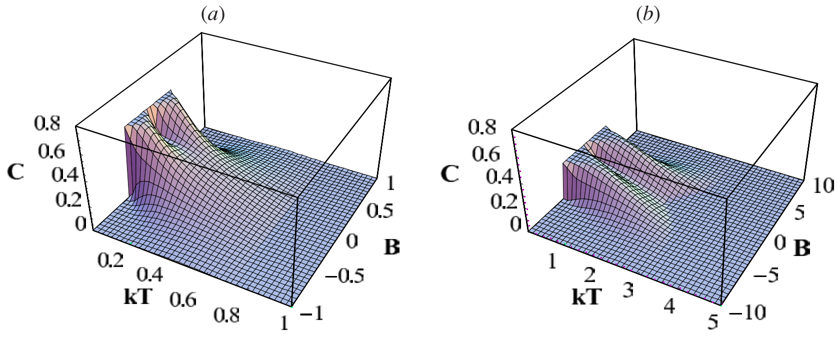


Figure 3. 3D plots of the thermal concurrence versus temperature T and the magnetic field B with $\delta = 1.4$ for the ferromagnetic and antiferromagnetic situations, respectively.

$$v_1 = \sum_{j=1}^{3} (\mathrm{e}^{-\beta E_j} + \mathrm{e}^{-\beta E_{j+3}}) B_j^2,$$
$$v_2 = \sum_{j=1}^{3} (\mathrm{e}^{-\beta E_j} + \mathrm{e}^{-\beta E_{j+3}}) C_j^2,$$
$$w = \sum_{j=1}^{3} (\mathrm{e}^{-\beta E_j} + \mathrm{e}^{-\beta E_{j+3}}) B_j C_j, \tag{24}$$

where the property $B_j C_j = A_{j+3} B_{j+3}$ for $j = 1, 2, 3$ has been used. Furthermore, it can be proved with easy that $|w| \leqslant \sqrt{v_1 v_2}$. Then the thermal concurrence of the system is written by

$$C = \frac{2}{Z} \max\{|w| - \sqrt{u_1 u_2}, 0\}. \tag{25}$$

The temperature T and magnetic field B dependences of the thermal concurrence C are presented in figure 3 for $J = \mp 1$, respectively. It is shown that the thermal entanglement in the ferromagnetic case is more sensitive to the temperature than that in antiferromagnetic case. The higher concurrence occurs at extremely low temperatures for the ferromagnetic situation. In figure 4, we show the thermal entanglement of the model as a function of the zero-field splitting parameter δ of the magnetic field and the temperature T. Obviously the thermal concurrence is controlled by the applied field at finite temperature, and the stronger entanglement can be obtained by varying the zero-splitting magnetic field parameter. The threshold temperature of the model versus the magnetic field is diagrammatized in figure 5 for the different B field, the splitting parameter δ and the coupling strength J. We note that the threshold temperature is not sensitive to the inhomogeneity of the applied field for the ferromagnetic case.

We know that the probability distribution P of the eigenstates in the thermal state χ is $P_i = Tr\{|\phi_i\rangle\langle\phi_i|\rho\} = \frac{\mathrm{e}^{-\beta E_i}}{Z}$ with $i = 0, 1, 2, 3, \ldots, 7$. The magnetization M_z per spin of the system in the z direction is $M_z\left(=\frac{2\langle S_z\rangle}{N}\right) = \frac{2}{N}\sum_{i=0}^{7}\langle S_z\rangle_i P_i$. It gives that

$$M_z = \frac{1}{3}\,\frac{3\mathrm{e}^{-\beta E_0} + \mathrm{e}^{-\beta E_1} + \mathrm{e}^{-\beta E_2} + \mathrm{e}^{-\beta E_3} - \mathrm{e}^{-\beta E_4} - \mathrm{e}^{-\beta E_5} - \mathrm{e}^{-\beta E_6} - 3\mathrm{e}^{-\beta E_7}}{\mathrm{e}^{-\beta E_0} + \mathrm{e}^{-\beta E_1} + \mathrm{e}^{-\beta E_2} + \mathrm{e}^{-\beta E_3} + \mathrm{e}^{-\beta E_4} + \mathrm{e}^{-\beta E_5} + \mathrm{e}^{-\beta E_6} + \mathrm{e}^{-\beta E_7}}.$$

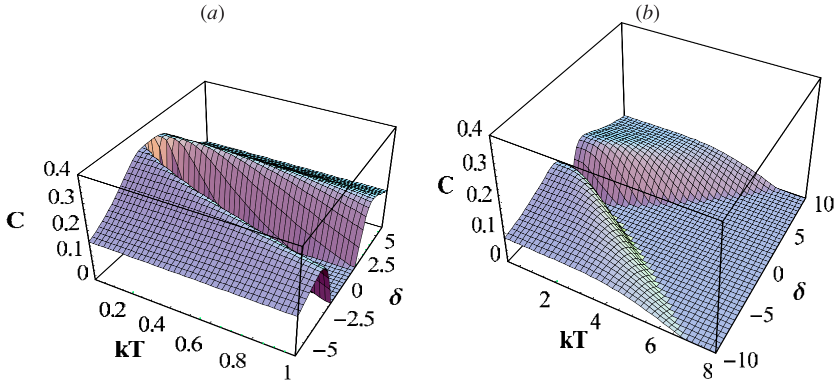


Figure 4. 3D plots of the thermal concurrence versus temperature T and the zero-field splitting parameter δ with $B = 0$ for the ferromagnetic ($J = -1$) and antiferromagnetic ($J = 1$) situations, respectively.

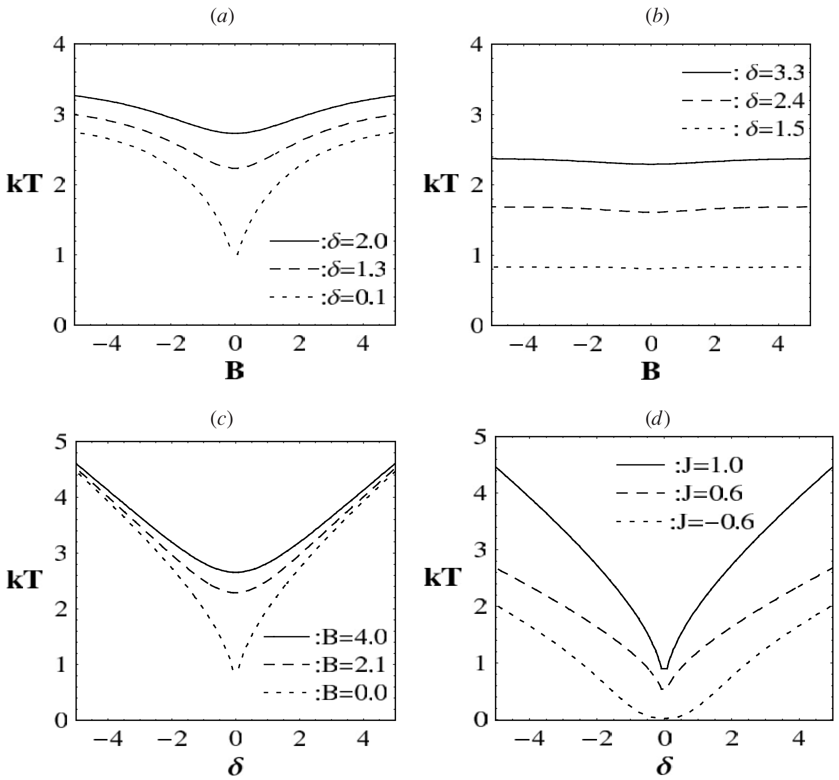


Figure 5. The variations of the threshold temperature with the magnetic field B and the inhomogeneity of the field in the ferromagnetic and antiferromagnetic cases: (a) $J = 1$ and $\delta = 0.1$ (dotted), 1.3 (dashed), 2.0 (solid); (b) $J = -1$ and $\delta = 1.5$ (dotted), 2.4 (dashed), 3.3 (solid), respectively; (c) $J = 1$ and $B = 0$ (dotted), 2.1 (dashed), 4.0 (solid), respectively; (d) $B = 0$ and $J = -0.6$ (dotted), 0.6 (dashed), 1.0 (solid), respectively.

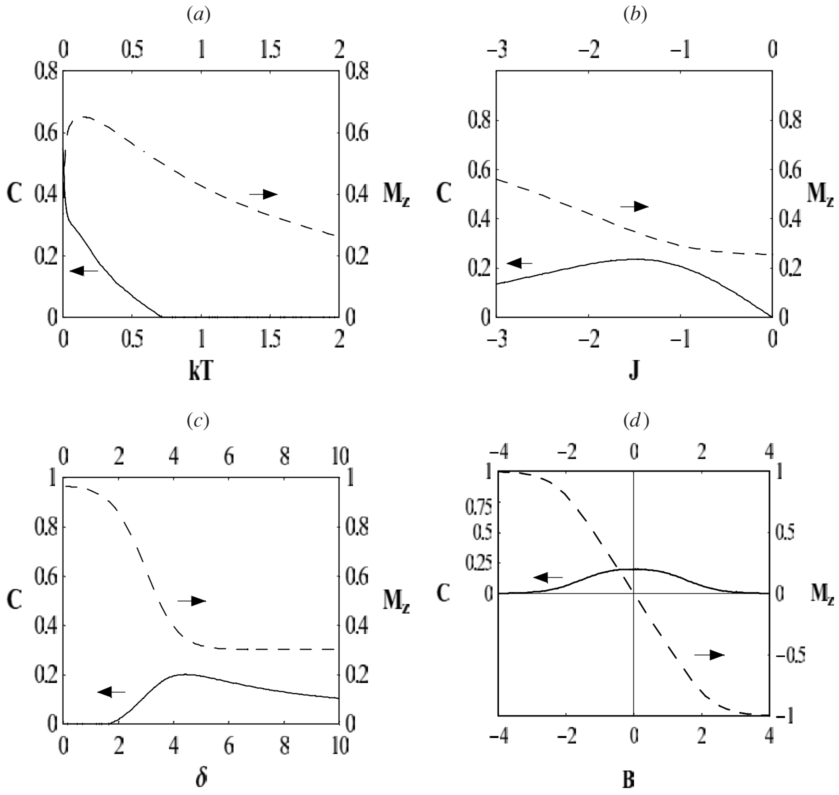


Figure 6. The thermal concurrence (solid) and the magnetization (dashed) as a function of the inhomogeneity parameter δ and the magnetic field B for the ferromagnetic case: (*a*) $J = -1$, $B = -0.4$, $\delta = 1.4$; (*b*) $B = -1$, $kT = 1$, $b = 4$; (*c*) $J = -1$, $kT = 1$, $B = -1.5$; (*d*) $J = -1$, $kT = 1$, $\delta = 3$.

Figure 6 shows the variations of the magnetization and the thermal concurrence as functions of the temperature, coupling strength, zero-field splitting parameter and magnetic field, respectively, for the ferromagnetic case. It shows that the thermal entanglement has the completely different property with the magnetization for the ferromagnetic case. The antiferromagnetic case is presented in figure 7. The thermal concurrence decreases when the temperature increases for both of the ferromagnetic and antiferromagnetic cases, although the magnetization is non-monotonous via the temperature for the ferromagnetic case (see figures 6(*a*) and 7(*a*)). Figures 6(*b*) and 7(*b*) show that the thermal concurrence is lower when the strength of the coupling is near zero, but the magnetization of the system can be finite. We find that the zero-field splitting parameter can adjust the thermal entanglement more effectively than the applied field B (see figures 6(*c*), (*d*) and 7(*c*), (*d*)), although the magnetization of the system is insensitive related to this splitting parameter for the antiferromagnetic case as shown in figure 7(*c*). It is very interesting to observe that the entanglement can describe the quantum correlation of the spin chain, which is independent of the magnetization of the spin system. In figure 8, we show the difference of the thermal entanglements between the three-qubit and two-qubit systems. Very recently, it has been proposed that the two-qubit system can be entangled in distinct temperature regions [16]. Here, in the three-qubit system

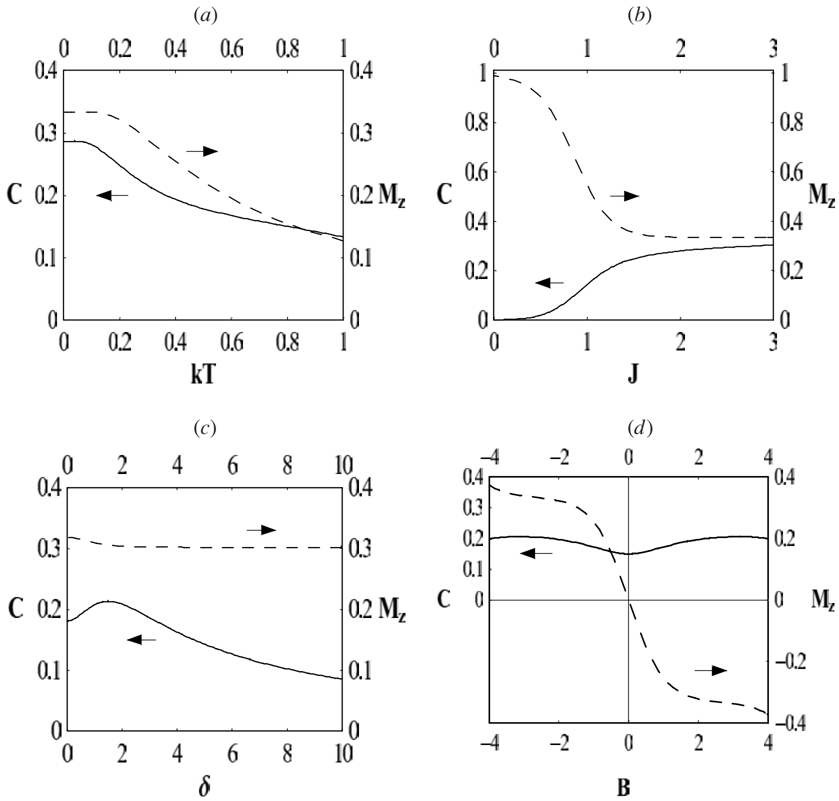


Figure 7. Comparison of the thermal concurrence (solid) and the magnetization (dashed) in the antiferromagnetic case: (*a*) $J = 1$, $B = -0.4$, $\delta = 1.4$; (*b*) $B = -5$, $kT = 1$, $b = 3$; (*c*) $J = 1$, $kT = 1$, $B = -1.5$; (*d*) $J = 1$, $kT = 1$, $\delta = 3$.

with the nonuniform applied field, the effect of the temperature is also very sensitive for the entanglement at extremely low temperature. In figure 9, we present the thermal concurrence with the entangled and unentangled ground states. When $J = -1$, $\xi = 1.5$ and $B = -0.8$, the ground state of the system is the unentangled state $|\psi_0\rangle$. But the thermal entanglement of the system is nonzero when temperatures are lower than the threshold one $kT_{\mathrm{th}} = 1.2$, as is shown in the solid in figure 9(*a*). As the magnetic field $B = -0.7$, the ground state of the system becomes as the entangled state $|\psi_1\rangle$, the thermal concurrence decreases monotonously when temperatures increase, as shown in the dotted line in figure 9(*a*). For the antiferromagnetic case ($J = 1$), as $\xi = 1.5$ and $B = 4.5$, the ground state of the system is the unentangled state $|\psi_7\rangle$. But the thermal state of the system is entangled as temperatures are below the threshold one $kT_{\mathrm{th}} = 3.2$, as shown in the solid in figure 9(*b*). As the applied field $B = 4.4$, the ground state is the entangled $|\psi_4\rangle$. In this case, the thermal concurrence decreases monotonously when temperatures increase, as shown in the dotted line in figure 9(*b*). That is, figures 9(*a*) and (*b*) provide us a fact that (i) the ground state is entangled and its thermal state is also entangled when the temperature is lower than the threshold temperature; (ii) the ground state is unentangled, but the corresponding thermal state is entangled as temperatures are below the threshold temperature. It shows the very interesting features of the entanglement of the qubits system.

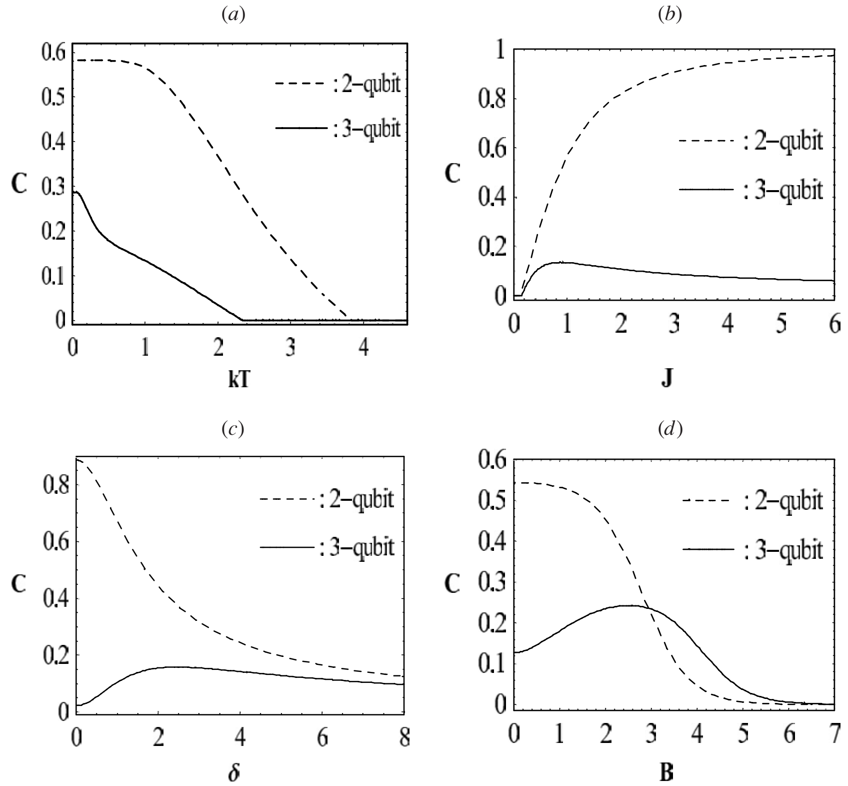


Figure 8. Comparison of the concurrences between the three-qubit (solid) and two-qubit (dashed) systems: (*a*) $J = 1, B = 0.4, \delta = 1.4$; (*b*) $B = 0.4, kT = 1, b = 1.4$; (*c*) $J = 1, kT = 1, B = 0.4$; (*d*) $J = 1, kT = 1, \delta = 1.5$.

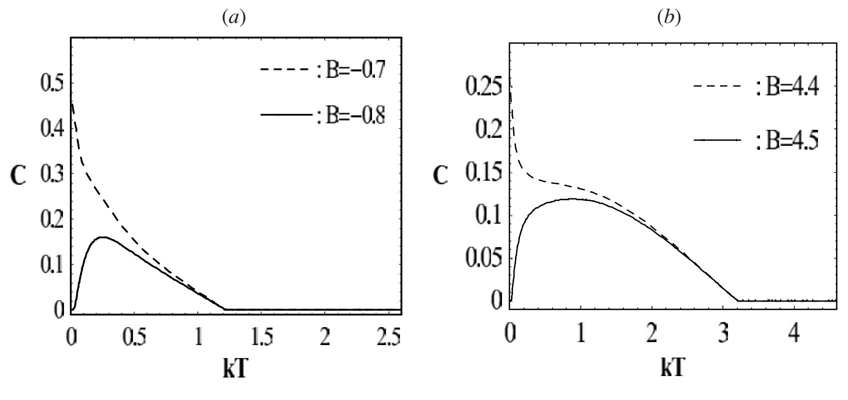


Figure 9. The thermal entanglement of the three-qubit system with the entangled and the untangled ground states, respectively, with $\xi = 1.5$: (*a*) $J = -1$, $B = -0.8$ (solid), and $B = -0.7$ (dotted); (*b*) $J = 1$, $B = 4.5$ (solid), and $B = 4.4$ (dotted).

4. Conclusion

We have studied the quantum entanglement of the energy eigenstates and the thermal entanglement of a three-qubit system by considering the pairwise concurrence in the presence of an inhomogeneous magnetic field. The threshold temperature of the thermal entanglement is discussed, and the comparison of the concurrence between the three-qubit and two-qubit systems is given. Our results show that the pairwise entanglement is dependent on the number of the qubits, and the coupling of the other qubit to the pair can increase the pairwise entanglement. This means a kind of the quantum correlation exists in a Heisenberg spin model.

The ground state of the system is either entangled or unentangled depending on the applied magnetic field and the coupling strength between the qubits. The critical magnetic fields exist and the ground state of the three-qubit system is divided into the four cases: the two entangled states and the two unentangled states for both the ferromagnetic and antiferromagnetic situations. We observe that the thermal entanglement of the system exists even if the ground state is separable, when the temperature is below threshold one. This means that the entanglement of the system can be increased as the temperature increases, although the entanglement is often a low-temperature phenomenon. It would be very interesting and possible to extent our investigation to large objects, especially various physical systems, such as a semiconductor, a superconductor and a Kondo model, etc. Recently, a method in which the entanglement of any pure quantum state can be experimentally determined is proposed based on a simple projective measurement when the state is available in a two-fold copy [20, 22]. Nearly all protocols requiring shared quantum information rely on the quantum entanglement. It is known that creating entanglement of many qubits is still a great challenge theoretically and experimentally. The cluster states, one kind of multipartite entangled states, can be created by an Ising-type interaction [5, 23], and the quantum computer can be realized based on these cluster states. We hope the above discussions would be helpful for the procedure to create the multipartite entanglement in quantum systems.

Acknowledgments

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